

Taking $N \rightarrow 0$ with S matrices

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Abstract

Interesting physical results can be obtained from sigma models by taking the number of fields N to zero. I discuss how one can make sense of this limit by using exact S matrix techniques. I review how this can be done for the case of self-avoiding polymers, and speculate on the application to the replica limit of disordered systems.

1 Introduction

Significant progress has been made on understanding the phase structure of disordered models in two dimensions. One key tool has been in formulating the systems in terms of replica or supersymmetric sigma models. A sigma model is a field theory where the fields take values on a manifold. Models can be classified according to their symmetry [1], so systems with very different physical origins can still be described by the same sigma model.

The phase structures of many two-dimensional sigma models are fairly well understood, so we know a great deal about the phases of the corresponding disordered models (although the latter seems to exhibit even richer behavior; see for example the discussion in [2]). The types of critical points possible have been extensively studied [3]. Much less progress has been made on understanding the non-trivial critical points themselves. These points should have conformal invariance and thus be described by conformal field theories. However, only in a few cases [4, 5] has a non-trivial critical point been definitively identified as a conformal field theory, and critical exponents computed exactly.

The supersymmetric sigma models describing disordered systems are well-defined. Unfortunately, many of the conventional theoretical approaches to such problems are either extremely difficult or impossible, because the bosonic part of the field manifold is non-compact. The replica sigma models can be formulated on compact manifolds. The (huge) disadvantage here is that one must take the number of fields (the dimension of the field manifold) to zero at the end of computation. This is at best ill-defined outside of perturbation theory, and at worst completely wrong.

There is, however, one well-understood field theory where the number of fields is taken to zero. This is a self-avoiding random walk (often known as a polymer) in two dimensions. The field-theory description of this problem involves a field transforming as the vector of $O(N)$, and at the end, one must send N to zero. There are formulations of the problem which are completely well-defined as a function of N , and which are believed to be in the same universality class. Namely, there is a lattice model defined for all N , not just integer values [6]. This continuum limit of this lattice model for any N can be mapped onto a field theory by Coulomb-gas techniques. The mapping is not rigorous, but widely believed to give a field theory in the correct universality class. Moreover, one can extend these results even further by finding a description of the polymer field theory in terms of exact S matrices [7]. Again, this description is precisely defined for any N , and one only need make the assumption that the models are in the same universality class.

The aim of this contribution to the conference proceedings is to explain the last paragraph in more detail, and to present some speculations on the application to one particular type of system, the metal-insulator transition critical point in the GSE class. This class is interesting in that it is the two-dimensional version of one of the original random matrix ensembles of Wigner and Dyson. Since this class of models is integrable for finite integer N [8, 9], one might hope to be able to apply S matrix methods here as well. One key problem I will discuss is the singularity that appears here and in the polymer model as one decreases the number of fields through $N = 2$.

In section 2 I review the arguments for the existence of a metal-insulator transition in the GSE class. In section 3, I review the solution of the dilute/dense transition for a self-avoiding polymer. I show how this problem can be solved by applying S matrix methods. In section 4, I speculate on the applications of S matrix methods to the metal-insulator transition in the GSE class.

2 The metal/insulator transition in the GSE class replica sigma model

The replica sigma model for the GSE class in two dimensions is a sigma model where the field takes values in the manifold

$$\frac{O(2P)}{O(P) \times O(P)}.$$

For $P > 1$, the exact S matrix for this sigma model has been found [8, 9]. The action can be written in terms of a $2P \times 2P$ matrix field Φ :

$$S = \frac{1}{2g} \text{tr} \int d^2x \partial^\mu \Phi^\dagger \partial_\mu \Phi \quad (1)$$

where Φ must be real, symmetric, orthogonal and traceless. The beta function for this model has been computed to at least three loops [10]. For the more general sigma model on the manifold $O(N)/O(N-P) \times O(P)$ the beta function at two loops is

$$\beta(g) = -(N-2)g^2 - [2P(N-P) - N]g^3 \quad (2)$$

A number of important facts are apparent from this beta function. First of all, there is a critical point at $g = 0$. This is the trivial fixed point, where the sigma model manifold effectively becomes flat and the model reduces to free bosonic fields. A crucial consequence of this beta function is that for $N > 2$ ($P > 1$ for the GSE case), is that the trivial fixed point is unstable. At longer distance, the system flows away from the trivial fixed point and the coupling g increases. Another

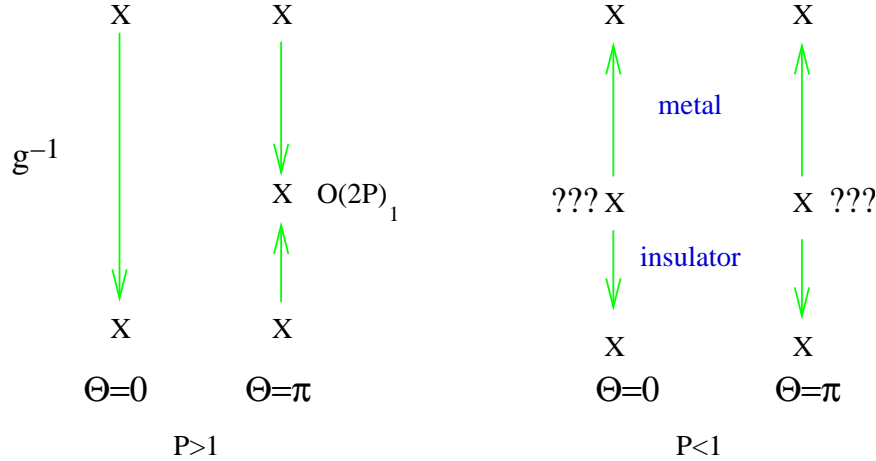


Figure 1: The proposed phase diagram for the $O(2P)/O(P) \times O(P)$ sigma model

consequence of (2) is that there is no evidence for a zero of $\beta(g)$ other than the trivial fixed point for any $N \geq 0$. This of course does not prove that there is no such fixed point, only that it can not be found by a perturbative expansion valid near the trivial fixed point.

The coupling g increasing arbitrarily as the distance scale is increased is a signal of Anderson localization, because $g \rightarrow \infty$ corresponds to the conductance decreasing to zero. However, this is **not** what happens in the GSE class. The reason is that $\beta(g)$ for g small changes sign when N is decreased below 2. This means that the $g = 0$ trivial fixed point is **stable** when $N < 2$, and so there is a *metallic phase* in the disordered system for small enough g . For large enough g , one expects disorder to dominate and localization to occur. This implies there is a metal/insulator transition at some value g_c . Since no fixed point appears in the beta function, the value of g_c must be out of the regime of validity of perturbation theory. These considerations lead to the phase diagrams for the $O(2P)/O(P) \times O(P)$ sigma model at $\theta = 0$ displayed in the figure 1. The left one is valid for $N > 2$, while the right applies to $N < 2$, and includes the replica limit of the GSE class.

A parameter called the θ angle appears in some sigma models. This is crucial for example in the replica approach to the plateau transition in the quantum Hall effect [11]. The theta angle is associated with field configurations called instantons. This is well known for $P = 2$, where the $O(4)/O(2) \times O(2)$ sigma model reduces to two copies of the $O(3)/O(2)$ sigma model. In the $O(3)/O(2)$ model, the field takes values on the sphere, and is believed to describe the continuum limit of Heisenberg spin chains, with $\theta = 0$ describing integer-spin chains, while $\theta = \pi$ describes those with half-integer spin [12]. The field theory with $\theta = 0$ is gapped, and the spectrum is a triplet under the global $O(3)$ symmetry, while when $\theta = \pi$, the spectrum is gapless and forms doublets under the global symmetry [13]. For the GSE class model for general P , the parameter θ can take just two values $\theta = 0, \pi$. For $P > 2$, the behavior is similar to $P = 2$: the spectrum is gapped for $\theta = 0$ and gapless for $\theta = \pi$ [8].

The sigma model at $\theta = \pi$ with $P > 1$ therefore has a non-trivial fixed point when $\theta = \pi$, as illustrated in the left half of figure 1. Since the value of θ does not affect perturbation theory, the value of g_c must be outside the region of validity of perturbation theory. The question now is what happens for $P < 1$. For g small, perturbation theory is valid and the beta function (2) is applicable to both $\theta = 0$ and π . This means that the flow for $P < 1$ must be towards the trivial fixed point, at least for g small. One still expects that for g large, the model is in a disordered

phase. The simplest possible phase diagram for the $P < 1$ sigma model is shown in the right half of figure 1. The non-trivial critical point now describes a metal-insulator transition. Comparing the phase structure for $\theta = 0$ and π , an important distinction is that for the latter, the non-trivial critical point is present even for $P > 1$.

For $\theta = \pi$ and $P > 1$, the conformal field theory describing the critical point has been identified in [8]. It is the $O(2P)_1$ WZW theory, which essentially amounts to P free Dirac fermions. However, continuing this result to $P \rightarrow 0$ presents immediate problems. The vanishing of the beta function is not the only unusual thing happening at $N = 2$. I will show in section 4 the exact solution of [8, 9] becomes singular there, making taking the replica limit quite tricky. This is fortunate, because it does not seem likely that the metal-insulator transition in the GSE class is described by free fermions.

3 The S matrix approach to polymers

In the previous section I described how a “metal” phase appears in the field theories with $N < 2$. Even though the sigma models are solvable for $N > 2$, various pathologies happen to the solution as P is deformed to 1. This sort of behavior happens in all two-dimensional sigma models with $O(N)$ symmetry as N is decreased to 2. This shows that one cannot simply continue the results from $N > 2$ to $N \rightarrow 0$. Indeed, it even seems possible that results in such models valid for $N > 2$ are meaningless in the $N \rightarrow 0$ replica limit. However, one reliable prediction has been made already: the beta function computed for $N > 2$ implies a metallic phase in the replica limit. The existence of a metallic phase has been fairly well established for at least some models in class D , where the replica sigma model is $O(2P)/U(P)$ (although results for this model have numerous subtleties [2]).

The purpose of this section is to explain in detail a model with $O(N)$ symmetry where the results for $N > 2$ can be used to guide the way to $N \rightarrow 0$. This is the field theory describing self-avoiding random walks (often called polymers) [14]. The same pathologies that happen in the GSE class sigma model as $N \rightarrow 2$ also happen here. Nevertheless, the non-trivial fixed point when $N \rightarrow 0$ is fairly well understood [6]. It is important to emphasize that it is not possible to naively continue the results for $N > 2$ to $N \rightarrow 0$. What I will argue is that results known from solving the theory for $N > 2$ imply certain criteria that the $N \rightarrow 0$ theory must obey. These criteria allow one to solve the theory for $N < 2$.

The polymer critical point in two dimensions was first understood by using Coulomb-gas methods. There is another approach available: the S matrix approach to integrable models. Here one works directly in the continuum, but ends up reproducing and extending the results of the Coulomb-gas approach [7]. These sigma models are defined as classical two-dimensional field theories. In the S matrix approach, the model is treated as a $1 + 1$ dimensional quantum field theory, so the rotational invariance of the classical model turns into Lorentz invariance of the quantum model. All the states of this quantum theory can be described in terms of quasiparticles in a Hilbert space. One can in principle (and sometimes in practice) derive the masses and multiplicities of these quasiparticles from the Bethe ansatz, but it is usually easier to infer them from the symmetries of the theory. The quasiparticle scattering matrix for an integrable model must satisfy a number of extremely restrictive constraints. By requiring that it satisfy these constraints, and agree with the perturbative expansions and all the symmetries of the model at hand, one can infer the S matrix as well. Once the exact S matrix is known, essentially all thermodynamic quantities can usually be derived. In addition, geometrical quantities like the scaling function for the number of polymer configurations on a cylinder can be computed [15], as

well as some form factors [16].

One nice feature of the polymer problem is that there are several precise ways to define the model for all N . One way is via an $O(N)$ vector model on the honeycomb lattice. A fixed-length N -component vector \mathbf{n}_i obeying $\mathbf{n}_i \cdot \mathbf{n}_i = 1$ is placed on each site i of the lattice. The partition function of this two-dimensional classical model is taken to be

$$Z = \int [d\mathbf{n}_i] \prod_{\langle ij \rangle} (1 + K \mathbf{n}_i \cdot \mathbf{n}_j) \quad (3)$$

where $\langle ij \rangle$ are nearest neighbors. By construction this model has a global $O(N)$ symmetry. This choice of partition function should be in the same universality class as the usual vector model with energy $J \sum_{\langle ij \rangle} \mathbf{n}_i \cdot \mathbf{n}_j$. The reason for this choice is that this model is equivalent to a model which not only can be defined for any N , but which makes the relation to the polymer problem clear. By performing a high-temperature expansion, the action (3) is equivalent to a loop gas with action

$$Z = \sum K^{n_{links}} N^{n_{loops}} \quad (4)$$

The sum is over all closed self-avoiding and mutually-avoiding loop configurations on the hexagonal lattice, with n_{loops} the number of such loops, and n_{links} the number of links in these loops [6]. Here N appears in the partition function only as a parameter, and so can now be taken to be any value, not just the positive integers. In particular, when N approaches zero, the partition function is dominated by the configurations which have only a single loop. This loop is the self-avoiding polymer.

This model has a critical point at $K = K_c = (2 + \sqrt{2 - N})^{-1/2}$. This critical point is at real values of K only for $N \leq 2$. Thus the two-dimensional self-avoiding polymer $N \rightarrow 0$ has a non-trivial critical point separating the high-temperature ($K < K_c$) “dilute” phase from the low-temperature ($K > K_c$) “dense” phase. In the dense phase, the preferred polymer configurations cover essentially the entire plane. Exact critical exponents describing the dilute-dense transition were originally derived by applying Coulomb-gas methods to the action (4) [6].

When N is a positive integer, the continuum limit of the lattice model (3) should be described by the “ $O(N)$ ” non-linear sigma model, where the field manifold is $O(N)/O(N-1)$. This $O(N)$ -symmetric model is described by a real field $\vec{\phi}(x)$ with N components taking values on the $(N-1)$ -sphere, and so obeys $\vec{\phi} \cdot \vec{\phi} = 1$. It has action

$$S^{O(N)} = \frac{1}{g} \int d^2x \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \quad (5)$$

Note that weak coupling in the sigma model (g small) corresponds to K large in the lattice model. Hence the *dense* phase of the polymer model is analogous to the metal phase of the disordered model, while the dilute phase is analogous to the insulating phase.

The perturbative beta function (2) of this field theory for $N > 2$ indicates that the trivial fixed point $g = 0$ is unstable. There is no evidence for a non-trivial low-energy fixed point, from the beta function or otherwise (there are no instantons and hence no theta term for $N > 3$). Since the only fixed point is the trivial one, it is not obvious how to obtain any information about the dilute-dense transition by continuing the $N > 2$ results for the sigma model to $N \rightarrow 0$. Indeed, the S matrix has a singularity as $N \rightarrow 2$, and cannot be analytically continued past. Nevertheless, there is an S matrix describing the $O(N)$ model for $N < 2$ and in particular the dilute-dense transition as $N \rightarrow 0$.

The field theory (5) is integrable for any integer $N > 2$. The simplest ansatz is that there is a multiplet of N massive particles in the vector representation of $O(N)$ [17]. A hint in favor

of this ansatz is that the fields in the action (5) are in the vector representation of $O(N)$. To check the ansatz thoroughly, one needs to compute the S matrix for these particles. Because of the relativistic invariance of this theory, it is convenient to use the rapidity variable β_1 , defined in terms of the quasiparticle mass, energy and momentum as $E = m \cosh \beta_1$ $p = m \sinh \beta_1$. A two-particle state is $|i_1(\beta_1)i_2(\beta_2)\rangle$, where i_1 and i_2 label the member of the multiplet: $i_1, i_2 = 1 \dots N$. The two-particle S matrix $S_{i_1 i_2}^{j_1 j_2}(\beta)$ is the amplitude for scattering this initial state to the final state $|j_1(\beta_1)j_2(\beta_2)\rangle$. The integrability requires that the collision be completely elastic: the individual momenta do not change. Lorentz invariance requires that the S matrix element depends only on the difference of the momenta: $\beta = \beta_1 - \beta_2$. The $O(N)$ invariance requires that S be of the form

$$S_{i_1 i_2}^{j_1 j_2}(\beta) = S_0(\beta) \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + S_1(\beta) \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} + S_2(\beta) \delta_{i_1 i_2} \delta^{j_1 j_2} \quad (6)$$

for some functions $S_0(\beta)$, $S_1(\beta)$ and $S_2(\beta)$. In an integrable theory, the S matrix must satisfy the Yang-Baxter equation. This ensures that the multi-particle scattering amplitudes factorize into the sum of products of two-particle amplitudes, a requirement of integrability. The simplest solution of these constraints is [17]

$$\begin{aligned} \frac{S_1(\beta)}{S_0(\beta)} &= \frac{2\pi i}{(N-2)\beta} \\ \frac{S_2(\beta)}{S_0(\beta)} &= \frac{2\pi i}{(N-2)(i\pi - \beta)} \\ S_0(\beta) &= -\frac{\Gamma\left(1 + \frac{i\beta}{2\pi}\right) \Gamma\left(\frac{1}{2} - \frac{i\beta}{2\pi}\right) \Gamma\left(\frac{1}{N-2} - \frac{i\beta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{1}{N-2} + \frac{i\beta}{2\pi}\right)}{\Gamma\left(-\frac{i\beta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{i\beta}{2\pi}\right) \Gamma\left(1 + \frac{1}{N-2} + \frac{i\beta}{2\pi}\right) \Gamma\left(\frac{1}{2} + \frac{1}{N-2} - \frac{i\beta}{2\pi}\right)} \end{aligned} \quad (7)$$

A number of checks indicate that the entire spectrum of the sigma model consists of these N particles with this S matrix. It agrees with large- N computations [17]. By computing the energy at zero temperature in the presence of a magnetic field, one can obtain correctly the first terms of the beta function [18]. Another substantial check is to compute the free energy of this massive theory at non-zero temperature from this S matrix. The free energy in the weak-coupling limit must agree with that of $N-1$ free bosons (the action (5) as $g \rightarrow 0$), and indeed does [19]. These checks effectively prove the ansatz that (6, 7) are the exact S matrix for the $O(N)$ sigma model for $N > 2$.

This S matrix involves N particles, and so seems to require that N needs to be an integer to make sense. I will discuss below how to define an S matrix for continuous N , but to see what happens as $N \rightarrow 2$, it is useful to examine the zero-temperature energy in a magnetic field H . This is a standard computation in an integrable model with a $U(1)$ symmetry. At zero temperature, particles with rapidity $|\beta| < B$ fill a Fermi sea, where B depends on H . The energy of this sea is

$$E(H) - E(0) = -\frac{m}{2\pi} \int_{-B}^B d\beta \cosh \beta \epsilon(\beta) \quad (8)$$

where $\epsilon(\beta)$ is the energy lost by removing a particle from the sea. It is given by the equation

$$\epsilon(\beta) = H - m \cosh \beta + \int_{-B}^B d\beta' \kappa(\beta - \beta') \epsilon(\beta'). \quad (9)$$

where $\kappa(\beta)$ follows from the S matrix, and for this model is [18]

$$\kappa^{O(N)}(\beta) \equiv \frac{1}{2\pi i} \frac{d}{d\beta} \ln [S_0(\beta) + S_1(\beta)] \quad (10)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\beta} e^{-\pi|\omega|/(N-2)} \frac{\cosh\left(\frac{N\pi}{2(N-2)}|\omega|\right)}{\cosh\left(\frac{\pi}{2}\omega\right)} \quad (11)$$

The value of B is determined by imposing the condition $\epsilon(B) = 0$. Clearly these equations are valid and can be solved for continuous $N \geq 2$.

One interesting thing to note is that the above S matrix and energy do make sense right at $N = 2$. Moreover, it correctly describes the continuum limit of the lattice model (3,4) when $N = 2$. This is straightforward to check: when $N = 2$, the lattice model is equivalent to the classical XY rotor right at its Kosterlitz-Thouless point [6]. The appropriate field theory description is indeed a single boson perturbed by a marginally-relevant operator (the vortex creation operator). Explicitly, this field theory is the sine-Gordon model with action

$$S^{SG} = \int d^2x \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + G \cos(\sqrt{8\pi}\varphi) \right]. \quad (12)$$

Here the role of the coupling g is played by the coefficient G of the $\cos \beta_{SG} \varphi$ term. This point $\beta_{SG}^2 = 8\pi$ is where the sine-Gordon model closely resembles a sigma model: not only is the operator perturbing the trivial fixed point marginally relevant, but the model has a Lie-algebra symmetry. In fact, it turns out to have a symmetry larger than the $O(2)$ it satisfies by construction: the two particles form a doublet under a larger $SU(2)$ symmetry. In the sine-Gordon language, the particle doublet consists of a kink and an antikink in the field φ ; there are no other particles in the spectrum at this value of β_{SG} . In particular, there is no single-particle state corresponding to the field φ itself: φ creates only multi-particle states with the same number of kink as antikinks. Note that the action (5) with $N=2$ is **not** sufficient to describe this field theory: defining $\phi_1 = \cos[\sqrt{g/2}\varphi]$, $\phi_2 = \sin[\sqrt{g/2}\varphi]$ reproduces the first but not the second term in (12). For $N > 2$, one argues that (5) holds because there are no relevant terms one can add consistent with the $O(N)$ symmetry. However, as seen from (12), there indeed is such a term when $N = 2$. In this sense, the S matrix description is more closely related to the original non-perturbative lattice formulation than is the sigma model. In fact, note that this S matrix at $N = 2$ describes the physics near the *non-trivial* critical point at $K = K_c = 1/\sqrt{2}$, not the trivial fixed point at $K = \infty$. Thus continuing the $N > 2$ S matrix to $N = 2$ does give a sign of the new physics occurring for $N < 2$.

Although the S matrix is valid at $N = 2$, it cannot be continued past: a singularity at $N = 2$ is immediately apparent from the explicit form of the above S matrix. Such a singularity appears in all integrable $O(N)$ -invariant field theories in two dimensions. Continuing around the singularity leads to nonsensical results. For example, the kernel $\kappa^{O(N)}(\beta)$ makes no sense for $N < 2$, because the integral does not converge for real β . Since a naive continuation of the $N < 2$ S matrix to $N < 2$ is not possible, one must therefore formulate a different ansatz. Such an ansatz was given in [7]; I will present a slightly different argument for this result here. Based on what is known for $N \geq 2$, the following criteria for such an $N \leq 2$ S matrix seem reasonable:

1. The particles should be in the vector representation of $O(N)$.
2. The S matrix as $N \rightarrow 2$ from below should be the same as $N \rightarrow 2$ from above.

Note that the second criterion does not rule out a singularity at $N = 2$: derivatives of the S matrix elements with respect to N need not be continuous.

These criteria are sufficient to infer the $N < 2$ S matrix. The first criterion means that the S matrix must still be of the form (6). However, we must look for a solution of the Yang-Baxter

equation other than (7). Such a solution has long been known to exist [20, 7]. It is

$$S_0(\beta) = 0 \quad (13)$$

$$\frac{S_2(\beta)}{S_1(\beta)} = \frac{\sinh[\mu(i\pi - \beta)]}{\sinh[\mu\beta]} \quad (14)$$

where μ is defined by

$$N = 2 \cos(\pi\mu).$$

Crossing and unitarity yields

$$S_1(\beta) = \frac{\Gamma\left(\frac{1}{2} - \frac{\mu\beta}{i\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{\mu\beta}{i\pi}\right)} \prod_{k=1}^{\infty} \frac{\Gamma\left(2k\mu - \frac{\mu\beta}{i\pi}\right) \Gamma\left(1 + 2k\mu - \frac{\mu\beta}{i\pi}\right)}{\Gamma\left(2k\mu + \frac{\mu\beta}{i\pi}\right) \Gamma\left(1 + 2k\mu + \frac{\mu\beta}{i\pi}\right)} \\ \times \frac{\Gamma\left((2k-1)\mu + \frac{\mu\beta}{i\pi}\right) \Gamma\left(1 + (2k-1)\mu + \frac{\mu\beta}{i\pi}\right)}{\Gamma\left((2k-1)\mu - \frac{\mu\beta}{i\pi}\right) \Gamma\left(1 + (2k-1)\mu - \frac{\mu\beta}{i\pi}\right)}$$

This S matrix makes sense for $N \leq 2$; in fact it does not make sense when $N > 2$ because μ becomes imaginary. It agrees with the $N \geq 2$ S matrix when $N = 2$. It also has a nice intuitive interpretation: because S_0 vanishes, the world lines of the particles do not cross and one can think of them as the loops in the loop gas (4). These points led Zamolodchikov to conjecture that it describes the $O(N)$ lattice model for $N < 2$ [7] at the dilute/dense critical point and in the dilute phase.

This conjecture was proven in [21]. A key element of the proof is that when one inserts the form (6) into the Yang-Baxter equation, the equation reduces to a set of functional relations relating S_0 , S_1 and S_2 . These relations involve N only as a parameter, so the functions can be found for any N . This suggests there is an S matrix with identical physical properties valid at any N [21]. To find this equivalent S matrix, note that when $S_0(\beta) = 0$, the S matrix is

$$S = S_1(\beta)P + S_2(\beta)e,$$

where P and e are $N^2 \times N^2$ matrices. However, to compute quantities like the free energy, the explicit matrix form of P and e is not required. All that needs to be known is what algebra the matrices obey (for example, P is the permutation operator, and $e^2 = Ne$). In a different matrix representation of the same algebra, all physical consequences will be the same. In the polymer case, the algebra is called a Temperley-Lieb algebra, and representations exist for any $N < 2$ [21].

The S matrix approach therefore allows a precise *continuum* formulation of the model valid for any N on and off the critical point. Moreover, the critical limit ends up being equivalent to the Coulomb-gas description, which in the continuum is a free boson with a charge at infinity [22]. This proof means that all S matrix computations give identical results to those coming from the continuum Coulomb-gas description of the $O(N)$ model. In fact, since the S matrix description is valid non-perturbatively, one can compute new results such as the scaling function for the number of polymers on a cylinder [15] as well as correlators from the form-factor expansion [16].

The S matrix derived in [7] thus gives a natural extension of the $O(N)/O(N-1)$ sigma model to $N < 2$. It is motivated by the loop gas, but the above criteria for finding the $N < 2$ S matrix arise from studying only the $N > 2$ S matrix, not the lattice model. It thus gives hope that other $O(N)$ -invariant sigma models defined for $N > 2$ can be extended to $N < 2$.

4 Speculations on the GSE class

Unfortunately, what seems so clear in the polymer case is very muddy in the GSE case. I do not know of a lattice model in the same universality class. I also do not know of an S matrix for $P < 1$. In this section I discuss the situation, and one S matrix which seems plausible but which does not yield the phase diagram of section 2.

The massive $O(2P)/O(P) \times O(P)$ field theory at $\theta = 0$ and the massless one at $\theta = \pi$ are both integrable, and their exact spectrum and S matrices are known for $P > 1$ [8, 9]. A crucial fact is that both have kink states, and that at $P = 1$ and $P = 2$ these are the only states which remain in the spectrum. Thus to understand what happens for $P < 1$, one needs to concentrate on the kinks. The kinks in the $O(2P)/O(P) \times O(P)$ sigma model at $\theta = \pi$ are in the two spinor representations of $O(2P)$ [8], each of dimension 2^{P-1} . At $\theta = 0$, they are in the representations with highest weight $2\mu_s$ and $2\mu_{\bar{s}}$ (where μ_s and $\mu_{\bar{s}}$ are the highest weights of the spinor representations). These can be obtained by taking the symmetric tensor product of two s and two \bar{s} representations respectively, and each is of dimension $(2P-1)!/P!(P-1)!$ [8, 9].

The complete explicit form of the kink S matrix for $P \geq 1$ is quite complicated and not particularly illuminating. It does enable one to find the kernel $\kappa(\beta)$ in the equations for the energy in a background magnetic field, as described in (8,9). The magnetic field can be chosen so that the Fermi sea is filled with only one kind of kink, corresponding to the state with weight equal to the highest weight $2\mu_s$. The resulting kernel for the sigma model at $\theta = 0$ is

$$\kappa^{sigma}(\beta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\beta} \left[1 - e^{-\pi|\omega|/(2P-2)} \frac{\sinh\left(\frac{P\pi}{(2P-2)}|\omega|\right)}{\cosh\left(\frac{\pi}{2}\omega\right)} \right]. \quad (15)$$

This kernel cannot be continued naively to $P < 1$. For the sigma model at $\theta = \pi$, the equations (8) and (9) are modified because the particles are massless, but as described in detail in [8], the most important kernel is the same as at $\theta = 0$.

As with the polymers, it is important to understand what happens at the singularity, here $P = 1$. Unlike the polymers, there is no known underlying lattice model to appeal to for guidance. However, the polymer case taught a valuable lesson in that the $N > 2$ S matrix was still valid for $N = 2$. Even more remarkably, it described the physics in the region around the non-trivial fixed point which just appeared at $N = 2$. This of course may have just been a stroke of luck peculiar to that problem. But since the action of the sigma model at $P = 1$ reduces to that of a free boson, the S matrix is the main piece of information potentially useful here. Thus to proceed it is necessary to assume that it is meaningful to continue the spectrum and S matrix for $P > 1$ to $P = 1$.

The first thing to note is that for $P = 1$ the dimensions of the representations with highest weights $\mu_s, \mu_{\bar{s}}, 2\mu_s$ and $2\mu_{\bar{s}}$ are each 1. Thus when $P = 1$ there should be two particles. The $P \rightarrow 1$ limit of (15) is equal to the $N \rightarrow 2$ limit of (11). Thus

$$S_{ss} = S_1 + S_2 = \frac{\Gamma\left(1 - \frac{\beta}{2\pi i}\right) \Gamma\left(\frac{1}{2} + \frac{\beta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\beta}{2\pi i}\right) \Gamma\left(\frac{1}{2} - \frac{\beta}{2\pi i}\right)}$$

here. For $N = 2$ in the $O(N)$ model, backscattering between the two particles is allowed, because they came from the same multiplet in the vector of $O(N)$. For $P > 1$ in the sigma model, backscattering between s particles and \bar{s} is forbidden, but the extra conservation law requiring this reduces to ordinary energy conservation when $P = 1$. Thus when $P = 1$, the one s particle

and the one \bar{s} can backscatter. The correspondence of this S matrix element and the possibility of backscattering means that it is plausible that the sigma model at $\theta = 0$ is identical to the $N = 2$ $O(N)$ model, namely the sine-Gordon model (12) at $\beta_{SG}^2 = 8\pi$.

Now comes the speculating. To proceed, one needs to impose criteria for a $P < 1$ S matrix. A reasonable set of criteria for $\theta = 0$ is

1. The massive particles for $P < 1$ are in the representations of $O(2P)$ with highest weights $2\mu_s$ and $2\mu_{\bar{s}}$.
2. Backscattering between the two representations occurs.
3. As $P \rightarrow 1$, the S matrix is that of the sine-Gordon model at $\beta_{SG}^2 = 8\pi$, which happens to be $SU(2)$ symmetric.

For $\theta = \pi$, the representations are instead the spinors s and \bar{s} .

The first criterion means that the S matrix describes the scattering of M particles, where $M = 2(2P - 1)!/P!(P - 1)!$ for $\theta = 0$, and $M = 2^P$ for $\theta = \pi$. The third means that for $P = 1$, the model is $SU(2)$ symmetric. The second criterion suggests that we group the particles in the s and \bar{s} representations into one representation of some larger symmetry. Since there are M particles, and $M = 2$ when $P = 1$, the simplest possibility seems to be that these particles are in the *vector* representation of $SU(M)$. Therefore an S matrix with particles in the vector of $SU(M)$ satisfies all three criteria. Such an S matrix satisfying all the constraints of integrability is well known, and describes scattering in the $SU(M)$ “chiral” Gross-Neveu model [23]. The critical point in this model is the $SU(M)_1$ conformal field theory. If this S matrix is the correct continuation to $P < 1$, then the metal-insulator transition is described by the conformal field theory $\lim_{M \rightarrow 1} SU(M)_1$. A simple but substantial check on the truth of this conjectured S matrix is to compute the central charge c of the conformal field theory describing the critical point. If the conformal field theory is to describe a phase transition in a disordered system, it must have $c = 0$. The central charge of $SU(M)_1$ is $c = M - 1$, so as $P \rightarrow 0$, $c \rightarrow 0$ as required. This is not an immediate consequence of the criteria imposed, so I view this as a good indication that the whole procedure is reasonable.

Unfortunately, this S matrix and conformal field theory do not seem to describe the physics of the 2d GSE metal-insulator transition. This is apparent from the behavior of the model away from the critical point. In the Gross-Neveu model, the operator perturbing the model away from the critical point is of dimension 2 (naively marginal). Computing the beta function to first non-trivial order yields that the operator is marginally relevant with one sign of coupling constant, and marginally irrelevant with the other sign. This contradicts the phase diagram in the right half of figure 1: at the non-trivial critical point the perturbing operator is relevant in both directions, not just one. Moreover, a perturbing operator of dimension 2 implies that the thermal exponent $\nu = \infty$. This is possible, but is not very consistent with the most recent numerical results, which suggest that ν is around 2.5 [24].

This is why I do not know what the appropriate S matrix is for this class of model, or for that matter, if one even exists. A key missing ingredient is a lattice model in the same universality class. The most likely candidate is a generalization of (3) with spins in the symmetric representation of $O(2N)$ instead of the vector, but I have not been able to develop a Coulomb-gas formulation for such a model. If this were to be accomplished, this would shed a great deal of light on the problem. Another interesting avenue to explore would be to understand for a given disordered system whether $\theta = 0$ or $\theta = \pi$. It seems likely that θ here is not a tunable

parameter, like the coefficient of the Wess-Zumino-Witten term appearing in other sigma models for disordered systems [25].

I should also note that an analogous continuation of the $O(N)$ Gross-Neveu model to $N < 2$ was proposed in [26]. This proposal involves particles in the vector representation of $O(N)$. Since particles in the vector representation of $O(N)$ do not appear in the Gross-Neveu model for $N = 3$ or 4, this proposal does not really obey the sort of criteria discussed in this paper. Nevertheless, there does not exist any better proposal yet, so the definitive answer for this situation as well as the GSE class remains unknown.

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